Chern-Simons contribution to the structure of the zero mode of the gauged nonlinear (2+1)-dimensional Schrödinger equation

L. A. Abramyan
Institute of Applied Physics, Russian Academy of Sciences, 46 Ul'yanov Street, 603000 Nizhny Novgorod, Russia

V. I. Berezhiani and A. P. Protogenov*
International Centre for Theoretical Physics, Trieste, Italy

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We study the solutions of the equations of motion in the model of gauged $(2+1)$-dimensional nonlinear Schrödinger equation. The contribution of Chern-Simons gauge fields leads to a significant decrease of the critical power of self-focusing. We also show that, for appropriate boundary conditions in the model considered, there exists a regime of turbulent motion of a hydrodynamic type. [S1063-651X(97)05610-9]

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I. INTRODUCTION

The nonlinear Schrödinger equation (NSE) is one of the basic models for nonlinear waves. The traditional field of application of the NSE has been nonlinear optics [1,2], where it describes the propagation of wave beams in nonlinear dispersive media. The NSE also arises in the treatment of various nonlinear waves in hydrodynamics and plasma physics [3]. A most important area of application in this case is the problem of the detailed description of collapsing field distributions [4–6]. With an opposite sign of the nonlinearity, the NSE is used as the basic model [7,8] of the low-dimensional field theory for describing vortices in the problem of Bose-Einstein condensation.

Recent interest in problems involving the solution of the NSE in spatially two dimensional (2D) systems has arisen in connection with the special properties exhibited by $(2+1)$D systems when the NSE is equipped with a gauge field through the replacement of the ordinary derivatives by covariant ones. In the infrared limit the main contribution to the equation of motion of the gauge field in a $(2+1)$D system is given by the Chern-Simons (CS) term within the system under consideration. For a certain relation of the coupling constants the contribution of the gauge field to the Hamiltonian compensates for the contribution from the nonlinearity. This leads to a soliton distribution of the field, which was found in Ref. [9]. The results of Ref. [9] have stimulated a number of papers [13–15] in this field.

The nature of this phenomenon can easily be understood if one takes into account that CS term breaks the $\mathcal{T}$- and $\mathcal{P}$-inversion symmetry of time and space. The chosen direction of the vector in the direction perpendicular to the plane can be considered as the chosen direction of rotation in the plane leading to the appearance of efficient repulsion. When this repulsion compensates for the attraction, the Hamiltonian turns out to be limited at the bottom, and the CS solitons [9] correspond to its zero value. These field distributions are solutions of the self-dual equations [10].

The results of Refs. [9,11,12] stimulated a number of papers in this field. We would like to focus on some of them. In Ref. [13] the structures of field configurations were analyzed in full detail for a nonlinearity in the NSE, which describes repulsion (in the absence of CS interaction) and takes into account the contribution of the nonzero mean value of the particle number density. Considering the initial problem, the authors of Ref. [14] concluded that at the most general initial condition (with a negative value of the Hamiltonian) the problems of the equations of motion for the NSE with the CS gauge fields correspond to the collapse regime. However, neither the spatial structure of the collapsing mode, nor the critical power for it (i.e., the number of particles in the mode) were analyzed in this paper. The problem of exact integration of the model under consideration was analyzed in Ref. [15]. The main result obtained is that the system cannot be integrated exactly excluding the following two cases: self-dual limit [9], and the situation when the $(2+1)$D equations can be reduced to the $(1+1)$D equations. An additional but still very important result of that paper was that the solitons of the gauged nonlinear Schrödinger equation (GNSE) have movable singularities on some curves in the two-dimensional plane. Detailed study of topological defects in low-dimensional systems have always been a key point for understanding dynamics of field distributions. The problem of so-called semi-local topological defects in the Chern-Simons-Higgs model was considered in a recent paper [16].

The distribution of complex-valued functions $\Psi(x,y,t)$ are defined on the manifold $\mathcal{M}$ which is multiconnected in spatial 2D systems. Therefore the fundamental homotopy group $\pi_1(\mathcal{M})$ determining analytical properties of function Ψ coincides with the braid group. There are actually several equivalent ways to reflect this fact in the theory. One of them is the Lagrangian approach that includes the effect of CS gauge fields into consideration. The CS term codes the existence and specific character of 2D point peculiarities contained in the Aharonov-Bohm gauge potentials within the long-wave description. One usually speaks of the long-range interaction represented by means of the CS gauge field as a statistical interaction between different field configurations.

*Permanent address: Institute of Applied Physics, Russian Academy of Sciences, 46 Ul'yanov Street, 603000 Nizhny Novgorod, Russia. Electronic address: alprot@appl.sci-nnov.ru

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The different representations induce the different forms of field distribution. There is the so-called anyon representation [17,18], when the gauge field in the explicit form is excluded from the Hamiltonian of the model, thus providing the representation of ‘‘noninteracting’’ (by means of the gauge field) configurations of field $\Psi(x,y,t)$. However, in this case the gauge field is proved to be included into the phase of function $\Psi(x,y,t)$ that contains a cut in the complex plane, which provides multivaluedness of the function.

The existence of a gauge interaction has an exclusively topological character, and is not associated with the quantum theory. This interaction, as a rule, was not taken into account when studying the classical dynamics of nonlinear models with a complex field in spatial 2D systems. Topological peculiarities certainly put additional restrictions on the quantization procedure in these systems [17,18]. The role of the CS gauge interaction in this case is to take into account *the vortex part of phase dynamics*, which was not usually considered in classical systems when the $(2+1)$D NSE model was used.

The purpose of this paper to study the equation of motion in the $(2+1)$D GNSE model. The main focus of attention is an investigation of the structure of the collapsing distribution of the fields. Specifically, by means of numerical integration of the equation of motion, we find the dependence of the critical power and of the effective width of the zero-energy mode on the coefficient k in front of the CS term. The limit $k\rightarrow\infty$, when interaction with the gauge field is negligible, may be used as a test. In this case the known values of the power and the width are restored.

If the phase of the field $\Psi(x,y,t)$ describes the longitudinal part in the gauge potential *completely*, the evolution of field configurations is determined *only by the temporal dependence of the gauge field*. We show that in this case the equations for the gauge field coincide with the equations of motion of an ideal fluid. The effects of the manifestation of the gauge field in classical systems with nontrivial topology, including the swimming motion at low Reynolds number within the $(2+1)$D hydrodynamics, are well known [19]. A new feature is the fact that the basis for the 2D turbulence in this case is chaotic dynamics of the CS gauge field. In this sense, the GNSE is a useful hydrodynamical tool [20].

One can see the following relation between the dynamics of the CS fields and the problem of 2D turbulence. It is well known that the CS action with appropriate boundary conditions is a way to classify conformal field theories [21]. The tools of the conformal field theory, in its turn, may be used [22] to study the 2D turbulence. We show that within the model under consideration the relation between the dynamics of CS fields and 2D turbulence problem may be stated beyond the application of the conformal field theory. This dependence can be represented considering the evolution of loops with the stochastization of lines near the points of the loop links.

The effect of contour links in terms of this paper reflects the effect of braiding world lines of the Aharonov-Bohm point singularities with formation of knots and links after projection of world lines onto the 2D space. Stochastization near the contour link points within formulation of 2D hydrodynamics of an ideal fluid in terms of contour variables [24,23] was discovered in Ref. [25]. Such a stochastic behavior has a universal character. It is based on existence of the braid group, and is closely connected with the arbitrary character of localization along the ‘‘time’’ axis of the point of world lines interlacing. Because of this the index k of the linking number proves to be a hidden parameter, which is not included into the Euler equations explicitly.

The paper is organized as follows. Section II contains the equation of motion for the CS gauge field and the GNSE for two different *Ansätze* corresponding to the goals of this paper. Section III is devoted to a numerical analysis of the problem. In conclusion, the results and open questions are discussed.

II. EQUATIONS OF MOTION

We consider a system with a Lagrangian density

$$\mathcal{L}=\frac{k}{2}\,\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma+i\Psi^*(\partial_t+iA_0)\Psi-\tfrac{1}{2}|(\boldsymbol{\nabla}-i\mathbf{A})\Psi|^2+\frac{g}{2}\,|\Psi|^4. \tag{1}$$

The equations of motion have the form

$$i\partial_t\Psi=-\tfrac{1}{2}(\boldsymbol{\nabla}-iA)^2\Psi+A_0\Psi-g|\Psi|^2\Psi, \tag{2}$$

$$[\boldsymbol{\nabla}\times\mathbf{A}]_\perp=-\frac{1}{k}\,|\Psi|^2, \tag{3}$$

$$\partial_t A_i+\partial_i A_0=-\frac{1}{k}\,\varepsilon_{ij}j_j\,. \tag{4}$$

Here g is the coupling constant and $j=\mathrm{Im}\Psi^*(\nabla-iA)\Psi$ is the current density. The Hamiltonian for Eq. (1) is

$$H=\frac{1}{2}\int d^2\mathbf{r}[|(\boldsymbol{\nabla}-i\mathbf{A})\Psi|^2-g|\Psi|^4], \tag{5}$$

where the potential A_μ, which is the auxiliary variable, is expressed in terms of $|\Psi|^2$ in the following way:

$$\mathbf{A}(r,t)=\frac{1}{k}\int d^2\mathbf{r}'\,\mathbf{G}(\mathbf{r}-\mathbf{r}')|\Psi|^2(r',t), \tag{6}$$

$$A_0(r,t)=\frac{1}{k}\int d^2\mathbf{r}'\,\mathbf{G}(r-r')\mathbf{j}(\mathbf{r}',t). \tag{7}$$

The Green function $\mathbf{G}(\mathbf{r})$

$$G_i(r)=\frac{1}{2\pi}\,\frac{\varepsilon_{ij}x_j}{r^2} \tag{8}$$

satisfies the equation

$$\boldsymbol{\nabla}\times\mathbf{G}(r)=-\delta^2(\mathbf{r}). \tag{9}$$

Since in the Hamiltonian formulation the potentials are uniquely determined by Eqs. (6) and (7), the gauge freedom

$$A_\mu\rightarrow A_\mu-\partial_\mu\varphi, \tag{10}$$

$$\Psi\rightarrow e^{i\varphi}\Psi \tag{11}$$

is fixed. This is achieved by using the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, with the boundary conditions

$$\lim_{r\to\infty} r^2 A_i(r,t)=\frac{1}{2\pi k}\,\varepsilon_{ij}x_j N, \tag{12}$$

$$\lim_{r\to\infty} A_0(r,t)=0. \tag{13}$$

The choice of the boundary condition (12) derives from the necessity of satisfying the integral representation of Gauss's law (3) of CS dynamics:

$$\Phi=\int d^2\mathbf{r}[\boldsymbol{\nabla}\times\mathbf{A}]_{\perp}=-\frac{1}{k}\int d^2\mathbf{r}|\Psi|^2. \tag{14}$$

Here the magnetic flux Φ and the number of particles

$$N=\int d^2\mathbf{r}|\Psi|^2 \tag{15}$$

are conserved giving the global constrain $\Phi=-(1/k)N$. In the result of Eqs. (2)–(4), there exists the continuity equation

$$\partial_t|\Psi|^2+\boldsymbol{\nabla}\cdot\mathbf{j}=0. \tag{16}$$

Let us use dimensionless fields and coordinates obtained by the following substitutions:

$$\Psi=|k|^{3/2}\rho e^{i\varphi},\quad A_0=-\frac{k^2}{2}\,w-\partial_t\varphi,$$

$$A_x=-ku+\partial_x\varphi,\quad A_y=-kv+\partial_y\varphi, \tag{17}$$

$$t\to\frac{-2}{k|k|}\,t,\quad x\to\frac{x}{|k|},\quad y\to\frac{y}{|k|}. \tag{18}$$

The equation of motion and the continuity equation, expressed in terms of the dimensionless fields $\rho\equiv\rho(x,y,t)$, $u\equiv u(x,y,t)$, $v\equiv v(x,y,t)$, and $w\equiv w(x,y,t)$ have the forms

$$\rho_{xx}+\rho_{yy}=-2C\rho^3-\rho w+\rho(u^2+v^2), \tag{19}$$

$$u_y-v_x=-\rho^2, \tag{20}$$

$$u_t-w_x=-2v\rho^2, \tag{21}$$

$$v_t-w_y=2u\rho^2, \tag{22}$$

$$\rho_t^2=2[(u\rho^2)_x+(v\rho^2)_y] \tag{23}$$

with the parameter $C=g|k|$ and notations $u_t=\partial_t u$, etc.

In the case of the usual NSE,

$$i\partial_t\Psi=-\boldsymbol{\nabla}^2\Psi-|\Psi|^2\Psi, \tag{24}$$

the substitution $\Psi=\rho e^{-i\varphi(x,y,t)}$ yields the equations

$$\rho_{xx}+\rho_{yy}=-\rho^3-\rho\varphi_t+\rho[(\varphi_x)^2+(\varphi_y)^2], \tag{25}$$

$$\rho_t^2=2[(\varphi_x\rho^2)_x+(\varphi_y\rho^2)_y]. \tag{26}$$

Comparing Eqs. (25) and (26) with Eqs. (19) and (23), we note the following distinctions. First, due to gauge invariance there are no derivatives of the phase φ in Eq. (19), as there are in Eq. (25). Their role is played by the gauge potentials. Therefore the evolution of the field $\rho(x,y,t)$ is defined by the time derivatives of the functions $u(x,y,t)$ and $v(x,y,t)$ in Eqs. (21) and (22). Unlike Eq. (25) the fields u and v are responsible for *the transverse* dynamics of the phase of the field Ψ. The *longitudinal* dynamics of the phase is described by the zero component $w(x,y,t)$ of the gauge potential which takes the place of the function φ_t in Eq. (19). The function $w(x,y,t)$ plays the role of a Lagrange multiplier, permitting one to take into account the restriction (20) of the Gauss law $\Phi=-(1/k)N$ locally.

Second, the continuity equation (23) can be obtained excluding the function w from Eqs. (21) and (22) with the aid of Eq. (20). This remark is associated with the following problem. Let us assume that in the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=-u_x-v_y+\Delta\varphi=0$, the phase φ satisfies the equation $\Delta\varphi=0$. Then the solution to the equation $u_x+v_y=0$ may be expressed in terms of a function $a(x,y,t)$ in the following way:

$$u=a_y,\quad v=-a_x. \tag{27}$$

In this case after replacing t by $-2t$, Eqs. (20) and (23) have the forms

$$a_{xx}+a_{yy}=-\rho^2, \tag{28}$$

$$\rho_t^2+u\rho_x^2+v\rho_y^2=0. \tag{29}$$

The set of Eqs. (28) and (29) represents the ''vorticity'' form of the Navier-Stokes equations (Euler equations) for two-dimensional flows of ideal incompressible fluids, where the function $a(x,y,t)$ has the meaning of a stream function. Note that hydrodynamic analogies have been used previously for the solution of $(1+1)$D NSE problem [26,20]. However, the present paper gives rigorous proof that the dynamics of the CS gauge field in the framework of the GNSE model (in the particular case of the Coulomb gauge with $\Delta\varphi=0$) is equivalent to the two-dimensional equation of motion of an ideal incompressible fluid. The remarkable fact is that there is a close analogy between the states with the constant flux in the turbulence and the CS anomaly [22] exposed by Eq. (28).

We pay our attention to one more circumstance. Using Gaussian law (28) we rewrite the particle number conservation law (29) for the vortex representation (27) in the following way:

$$\frac{\partial}{\partial t}\,\Delta a+\frac{D(\Delta a,a)}{D(x,y)}=0.$$

Here $D(\Delta a,a)/D(x,y)$ is the Jacobian. The dimensionless variables (18) in this representation make the CS coefficient k a hidden parameter which is not present explicitly in the last equation. However, the time and space coordinates, which are made dimensionless in such a way, are not equivalent. After transition $x\to x/|k|$, $y\to y/|k|$, $t\to t/|k|$ to the variables normalized by the coefficient k in the similar way, the term with the derivative $\partial/\partial t$ in the above equation acquires the coefficient $k/2$. This means that the classical limit $k\to\infty$ is equivalent to the transition $\Delta t\to 0$, i.e., to the transition to the static field distribution. Note that the characteristic rate of transition depends on the discrete index k of the loop linking number. The existence of hierarchy of discrete times $t_k = t/|k|$ is a common property of chiral CS systems. In this case its peculiarity is that it manifests itself already at classical level (in representation with equal scaling of the space and time coordinates with respect to k).

It is useful to compare the gauge invariance of the model and the Coulomb gauge used at $\Delta\varphi = 0$ with canonical transformations and with area-preserving transformations. The infinitesimal area-preserving diffeomorphism which acts in the frames of CS theory has the form

$$\xi_i \to \xi_i + \chi_i, \quad \partial_i \chi_i = 0, \tag{30}$$

where $\xi_i = (x,y)$ and $\chi_i = (A_1, A_2)$. The general solution of the equation $\partial_i \chi_i = 0$ is the sum of two terms,

$$\chi_i = \varepsilon_{ij} \partial_j a(\xi) + \sum_{k=1}^{b_1} c_k \chi_i^k, \tag{31}$$

where the second term describes the finite number (given by the first Betti number b_1) of harmonic forms on the two-dimensional phase space (A_x, A_y) of the CS theory. Diffeomorphisms which resulted from the first term in Eq. (31) are nothing but canonical transformations [27]. In the case when the phase space is a torus, the phase $\varphi(x,y,t)$, which satisfies the equation $\Delta\varphi = 0$, is a linear function, $\varphi = ax + by$. From the viewpoint of the NSE this corresponds to the constant direction of ray propagation assigned by the vector $\mathbf{n} \sim (a,b)$. In the general case of the phase space with arbitrary topology this equation does not hold, and the fact that the phase $\varphi(x,y,t)$ does not satisfy the equation $\Delta\varphi = 0$ gives rise to an ''additional'' longitudinal contribution to the potentials $u(x,y,t)$ and $v(x,y,t)$.

Let us consider for example the case when this phenomenon takes place. The *Ansatz* for the field $\Psi(x,y,t)$ corresponds to the generalized lens transformation [6,14]

$$\Psi(r,t) = \frac{\Phi(\boldsymbol{\zeta},\tau)}{g(\tau)} \exp[-ib(\tau)\boldsymbol{\zeta}^2/2 + i\lambda\tau]. \tag{32}$$

Here $\boldsymbol{\zeta} = \mathbf{r}/g(\tau)$, $\tau = \int_0^t du[f(u)]^{-2}$, and $b(\tau) = -f_t f = -g_\tau g$. The solution of the equation of motion (2) in this representation may be used as the initial data for Euler equation (29) (the continuity equation) written in the variables $\boldsymbol{\zeta}$ and τ. Below we focus our attention on the structure of the zero mode of Eq. (2).

The gauge potential transforms [9] upon the lens substitution as follows:

$$\mathbf{A}(r,t) \to [g(\tau)]^{-1} \mathbf{A}(\boldsymbol{\zeta},\tau), \tag{33}$$

$$A_0(r,t) \to [g(\tau)]^{-2} [A_0(\boldsymbol{\xi},\tau) - b(\tau)\boldsymbol{\zeta}\mathbf{A}(\boldsymbol{\zeta},\tau)], \tag{34}$$

while relations (6) and (7) are preserved, where the function $\rho = |\Phi|$. After these transformations Eq. (2) becomes

$$i\partial_r \Phi + (\beta\boldsymbol{\zeta}^2 - \lambda)\Phi = -\tfrac{1}{2}(\boldsymbol{\nabla} - i\mathbf{A})^2 \Phi + A_0 \Phi - g|\Phi|^2 \Phi, \tag{35}$$

because the function $\beta(\tau) = (b^2 + b_\tau)/2 = -f^3 f_{tt}/2$ does not equal zero in the case $\varphi(x,y,t) \sim b(x^2+y^2)$, $b(t) \neq t_0 - t$. However, if we are interested in collapsing solutions with [28,14] $f^2(t) \sim (t_0 - t)/\ln[\ln(t_0 - t)]$, the structure of the self-similar nonlinear core [14] of the solution is described by the solutions of the following equation:

$$-\lambda\Phi = -\tfrac{1}{2}(\boldsymbol{\nabla} - i\mathbf{A})^2 \Phi + A_0 \Phi - g|\Phi|^2 \Phi. \tag{36}$$

In Sec. III, by numerical calculation, we find the zero-energy localized ground state of the GNSE (36). We show the dependence of its effective width on the parameter $C = g|k|$ as well as the form of the functions u, v, and w.

III. SOLUTION STRUCTURE

For the numerical analysis of the solutions of Eq. (36) we use the method of the stabilizing multiplier [29]. The iteration approach for Eq. (36), which differs from Eq. (19) by the additional term $-\lambda\Phi$ on the left-hand side has the form

$$\Phi_{n+1} = M_n F^{-1}\{G(p)F[-2C\Phi_n^3 + j\Phi_n(u^2+v^2-w)_n]\}, \tag{37}$$

$$M_n = \left(\frac{\int d^2p(F\Phi_n)^2}{\int d^2p G(p) F\Phi_n F[-2C\Phi_n^3 + j\Phi_n(u^2+v^2-w)_n]} \right)^\alpha. \tag{38}$$

Here F (F^{-1}) are the operators of the direct (inverse) Fourier transform, $G(p) = -(p^2+\lambda)^{-1}$. The multiplier is $j=1$ or $j=0$, respectively, depending on whether the nonlinear contribution of the gauge field in Eq. (29) is taken into account or neglected. In the case $j=0$ the usual normalization in the NSE corresponds to $C = \frac{1}{2}$. Without loss of generality we shall suppose below that $\lambda = 1$. In the general case the relations between the function ρ, the parameter C, and the space scale L, referring to the arbitrary values of λ (denoted by a tilde) and at $\lambda = 1$ (denoted by a bar) are as follows: $\tilde{\rho}^2 = \sqrt{\lambda}\,\overline{\rho^2}$, $\tilde{C} = \sqrt{\lambda}\,\bar{C}$, and $\tilde{L} = \lambda^{-1}\bar{L}^2$. Therefore the following chain of relations takes place: $\tilde{N}_{j\neq 0} = \bar{N}_{j\neq 0}/\sqrt{\lambda} = \bar{N}_{j=0}/(2\sqrt{\lambda}\bar{C}) = \tilde{N}_{j=0}/(2\tilde{C})$.

We should choose the exponent α in the stabilizing multiplier M_n comparing the degrees of homogeneity of terms on the left- and right-hand sides of Eq. (36) proceeding from the requirements that $M_n \to 1$ at $n \to \infty$. Without the term $\Phi(u^2+v^2-w)$ the exponent α equals 3/2. New features of our problem are that the nonlinearity in Eq. (36) has the polynomial character of the type $-2C\Phi^3 + b\Phi^5$, because both of the terms Φw and $\Phi(u^2+v^2)$ are proportional to Φ^5. Therefore, for the convergence of the iteration approach

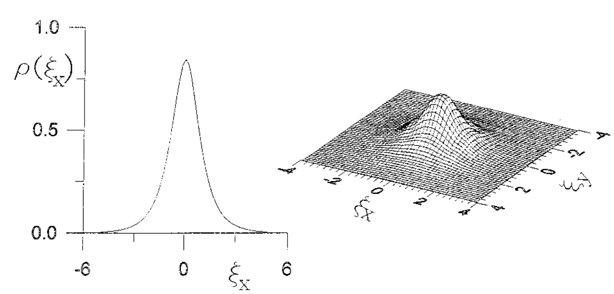

FIG. 1. Plot of the function $\rho(\zeta_x,0) = \rho(0,\zeta_y)$ and the surface $\rho(\zeta_x,\zeta_y)$.

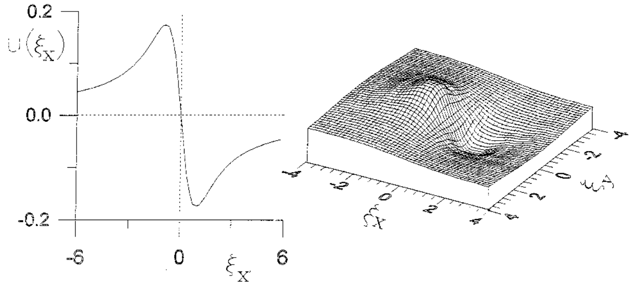

FIG. 2. Plot of the function $u(\zeta_x,0)$ and the surface $u(\zeta_x,\zeta_y)$.

α should belong to the range $\frac{5}{4}\leqslant\alpha\leqslant\frac{3}{2}$. In the simulation of the present paper we have used the value $\alpha=\frac{3}{2}$, which gives rapidly the value $M_n=1$ of the stabilizing multiplier. We have used the distribution of the form $\Phi(\zeta_x,\zeta_y)=(\sqrt{\gamma\delta}/\pi)\exp(-\gamma\zeta_x^2-\delta\zeta_y^2)$ as the initial field configurations with arbitrary constants γ and δ. In our calculations we have obtained rapidly the isotropic solutions.

To regularize the integrals (6) and (7) which diverge logarithmically in coincident points, we substituted $r^2\rightarrow r^2+\varepsilon^2$ into the expression for Green function (8) at numerical calculations of gauge potentials u and v. In the momentum space this corresponds to the substitution of $d^3p\ f(\varepsilon p)$ with $f(\varepsilon p)=\int_0^\infty dm\ \exp(-\sqrt{m^2+\varepsilon^2p^2})$ for d^3p. The factor $f(\varepsilon p)$ at $\varepsilon p\gg 1$ decreases exponentially, cutting off all momentum integrals. However, the infrared region remains the same, because at $\varepsilon p\ll 1$ $f(\varepsilon p)=1$. In our calculations the cutoff radius ε, which has the sense of the thickness of the vortex core, was equal to 10^{-2}.

The simulation was performed on a square lattice with linear sizes $L_x=L_y=12$. The maximum number of lattice sites was limited by the value $n=n_xn_y=128\times128$. To test our approach we used the solution of the equations of motion (36) with $A_\mu=0$ $(j=0)$ and with $C=\frac{1}{2}$ which gives the well-known value $N=11.703$, as well as the solution of a self-dual equation $\Delta\ln\rho=-\rho^2$ [9] when $N=4\pi\mathcal{N}$, $\mathcal{N}=1,2\ldots$ for $w=-\rho^2$, $u=\partial_y\ln\rho$, $v=-\partial_x\ln\rho$, and $C=1$.

Figures 1–3 show the configurations of the fields ρ, u, and w for the specific value of the parameter $C=4$. We may obtain the form of the function $v(\zeta_x,\zeta_y)$ using the relation $v(\zeta_x,\zeta_y)=-u(\zeta_y,\zeta_x)$. Using the function ρ obtained, we computed the dependence of the critical power N (the particle number) and of the effective width $\langle R^2\rangle=N^{-1}\int d^2\zeta\boldsymbol{\zeta}^2\rho^2(\boldsymbol{\zeta})$ on the parameter C. The results of calculations are given in Table I.

IV. CONCLUSION

In this paper we have studied the specific feature of the dimensionality of our problem reflecting the influence of the

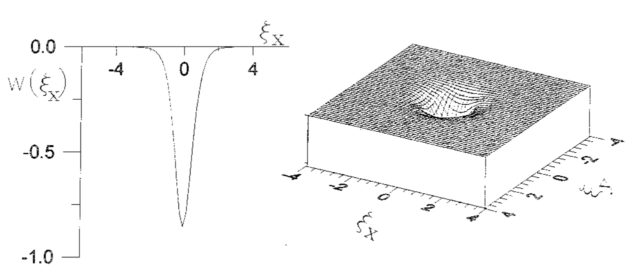

FIG. 3. Plot of $w(\zeta_x,0)=w(0,\zeta_y)$ and the surface $w(\zeta_x,\zeta_y)$.

TABLE I. Results of the calculations of the critical power N (the particle number), the effective width $\langle R^2\rangle$, for various values of the parameter C.

j	C	N	$\langle R^2\rangle$
0	0.5	11.703	1.2607
1	2.85	3.6483	1.2384
1	3	2.9216	1.2464
1	5	1.2825	1.2579
1	10	0.5973	1.2600
1	100	5.8528×10^{-2}	1.260 66
1	1000	5.8516×10^{-3}	1.260 66

CS gauge fields on field configurations in the GNSE model. Here we summarize some results.

It is seen from Eqs. (15)–(19) that if we neglect CS gauge fields [$j=0$ in Eq. (37)] the dependence of the particle number N on the parameter $C=g|k|$ has the form $N=N_0/C$. This dependence is shown by the dotted line in Fig. 4. It follows from the results shown in the first line of Table I that $N_0=5.585$. The contribution of the CS gauge fields ($j=1$ in Table I) leads to a sharp decrease in the values of N. In particular, $N_{j=1}(3)/N_{j=0}(0.5)\approx0.25$. The effective width $\langle R^2\rangle$ changes slightly. The calculated dependence $N(C)$ at $C\geqslant3$ is given in Fig. 4.

As expected, for a fixed value of the parameter C the number $N_{j=1}(C)$ is always greater than $N_{j=0}(C)$, because the CS gauge fields describe an effective repulsion. In the range $1<C<2.825$ we could not perform calculations in the framework of the method used, due to the breaking of convergence of the iteration methods (37) and (38). The reason is the change of the sign in the expression $-2C\Phi_n^3+j\Phi_n(u^2+v^2-w)_n$ at values $C\simeq1$ of the parameter C. This phenomenon shows that when the parameter C is of the order of unity the field contributions are characterized by a great (formally infinite) value of the flux $\Phi=\int d^2\mathbf{r}[\boldsymbol{\nabla}\times\mathbf{A}]_\perp$. Indeed, for self-dual configurations [9], when the field ρ decreases according to the power law $\rho(r)\propto r^{-(\mathcal{N}+1)}$, $\Phi=-4\pi\mathcal{N}/k$ with $\mathcal{N}=1,2,\ldots$. For the field distributions which we use, the decrease law is exponential. Roughly speaking, this corresponds to the values $\mathcal{N}\gg1$ of the desired limit of

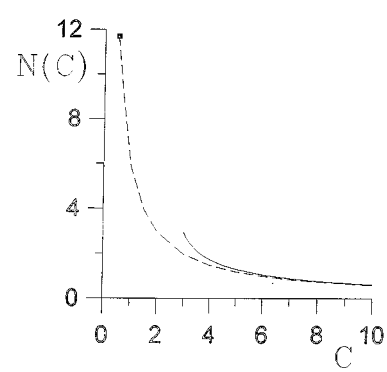

FIG. 4. Number of particles N as a function of the parameter C without taking into account the gauge field (dotted line) and with the gauge field (solid line). The point denotes the value $N(0.5)=11.703$.

the flux and the particle number $N=4\pi\mathcal{N}$ in the region $C \geqslant 1$.

The classical limit of the considered theory corresponds to the case $k\rightarrow\infty$, when the gauge field splits off from the field $\Psi(x,y,t)$. For Eq. (19) and for the value $N(C)$, the limit $C\gg1$ denotes that $N(C)\rightarrow 0$.

The present results correspond to the structure of the nonlinear core of the zero-energy localized ground state obtained by the lens transformation for the special value $\beta=0$ of the function $\beta(\tau)$ when $\beta(\tau)=1/(\tau_0+\tau)$. In this case the generalized lens transformation coincides with the conformal symmetry transformation [9] of the model. That is the reason why the form of the equation of motion (19) of the full model after the lens transformation coincides with Eq. (36). It will be very useful to compare the results obtained by the lens transformation for a finite function $\beta(\tau)$ in Eq. (35) with the results of a simulation using the full equations of motion (19)–(23) in the collapse regime. We plan to return to this problem in the framework of detailed simulation in the future.

Strong Langmuir turbulence in plasmas is usually described by the solutions of the NSE [Eq. (24)]. It is assumed that a cascade of randomly distributed self-similar collapsing fields is generated. In this paper we show that the specific features of spatially two-dimensional systems may lead to the traditional picture of turbulence associated with the Euler equations. However, for the hydrodynamic mechanism of turbulence (HMT) to be involved, it is necessary that a linear profile of the phase $\varphi(x,y)$ $(=ax+by)$ exists in each mode. This implies that the nonlinear (in x and y) contributions to the temporal evolution of the phase are small.

One of the media in which the HMT can play a role is an optical medium with random inhomogeneous guiding surfaces. Reflecting from the surfaces, the wave fronts acquire random directions of propagation. For media with weak Kerr nonlinearity, the nonlinear phase disturbance from adjacent points will not be important. The application of the HMT model suggested above requires separate consideration, and will be presented elsewhere.

In conclusion, we studied numerically the structure of the zero-energy collapsing mode in the GNSE model, observed a strong reduction of the critical power N in spatial two-dimensional systems as compared to the conventional values, and showed that in the case of appropriate boundary conditions the phenomenon of collapse inhibits the development of turbulence according to the hydrodynamic scenario.

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